On the behavior of bosonic systems in the presence of topology fluctuations

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Abstract

The behavior of bosonic systems in the presence of space-time foam is analyzed within the simplistic model of a set of scalar fields on a flat background. We discuss the formula for the path integral which allows to account for the all possible topologies of spacetime. We show that the proper path integral originates from the parastatistics suggested first by H.S. Green and that it defines a cutoff for the field theory.

1 Introduction

Spacetime foam is commonly believed to cure divergencies in particle physics [1] and therefore it will eventually allow to remove the unnatural and non-physical (and extremely restrictive) principle of the renormalizability of physical field theories[1]. However so far such a property has not been explicitly established yet.

At first glance the basic difficulty here stems from the problem of classifying topologies in 4-dimensions. Indeed, the adequate description of spacetime foam effects is reached in the euclidean quantum gravity advocated primary by S. W. Hawking [2] and developed by many authors (e.g., see Refs. [3]-[7]). The euclidean path integral for the expectation value of an observable B is

$$\langle B\rangle = \frac{\sum Be^{-S}}{\sum e^{-S}} \tag{1}$$

where S is the euclidean action and sum is taken over all field configurations and all topologies of the euclidean spacetime. The path integral is usually supposed to be taken in the two steps. First, one integrates over all field configurations keeping a specific topology fixed and then sums over different topologies, so that the partition function can be presented as

$$Z = \sum e^{-S} = \sum_{topologies} e^{-S_{eff}} \tag{2}$$

where S_{eff} is an independent effective action for each topology. Now one may use the semiclassical approximation (instantons) to evaluate contributions of different topological classes etc. and this is the way on which the further development of euclidean quantum gravity is going on (e.g., see Refs. [8] and references therein). We leave aside the loop quantum gravity [9], for essentials remain the same (as far as topologies is concerned).

It is clear however that results obtained on this way are rather restrictive in nature. Save the absence of an appropriate classification of different topologies, one can never justify that terms (topological classes) omitted give small effects. Even if such terms have bigger actions S_{eff} the number of such additional terms is enormous. One may think that the semiclassical approximation in (2) (though useful in investigating particular features) is not suitable. And indeed, if we believe that quantum gravity (i.e., topology fluctuations) provide quantum fields with a cutoff, then at very small scales (i.e., at very high energies) the physical space is effectively absent and all particular topologies may give only a negligible contribution to (2); for every term in (2) corresponds to a divergent quantum theory and as we shall see latter on (e.g., see (24)) smooth particular topologies are suppressed indeed.

In the present Letter we suggest absolutely equivalent formula for the path integral which allows to account for the all possible topologies of spacetime

[1] In particular, general relativity itself represents a non-renormalizable theory.

(even those which are apparently not smooth). As we shall see the proper path integral automatically defines a Lorentz invariant cutoff for the field theory as it was to be expected. The formula suggested follows quite naturally from the three well-established fundamental facts. 1) Any 4-dimensional manifold can be continued to the whole Euclidean space by adding non-physical regions of the space. Such a continuation is not unique however. In particular, the existence of a universal covering is the well-known mathematical fact. However in the general case the universal covering requires considering a curved space, while at high energies (at least at laboratory scales) the space looks to be flat. Our claim is that there always exists a continuation when the space remains to be flat (e.g., see Ref. [10]). 2) The discrepancy between the actual Green functions and those for the euclidean space is described by a topological bias of sources (i.e., the topology or the proper boundary conditions for the actual Green functions can be accounted for by additional sources). 3) The topological bias of sources has an equivalent description in terms of multi-valued fields. We stress that it is the basic fact which allows us to reformulate the sum over topologies in terms of the sum over multi-valued field configurations.

The first two facts represent the well known classical results. E.g., the universal covering (which is not more than the astrophysical way of the extrapolation of the laboratory coordinate system) and the concept of the topological bias were described in detail in Refs. [10, 11]. In particular, in astrophysics when we look at the sky we always have deal with the universal covering and this allows to give the most natural explanation for the all the variety of the observed dark matter phenomena (see the above papers and Ref. [12] where theoretical rotation curves for spiral galaxies were shown to be in a very good agreement with observations). The bias of sources and the "standard" continuation (i.e., without introducing a non-flat metric) is the standard tool for solving different electrostatic problems in classical electrodynamics (e.g., see the image method in Ref. [13]).

The last fact (the multi-valued nature of fields) being transparent is however less known. The basic construction was suggested in Ref. [14] and developed in Refs. [15, 16]. We stress that the fact that any particular topology admits an equivalent description in terms of multi-valued fields was proved in Ref. [14]. It turns out that multi-valued fields have the realization in terms of the so-called generalized statistics suggested first by H.S. Green in the attempt to solve the problem of mathematical inconsistencies (renormalization and regularization procedures) in quantum field theories. We also point out that in fact multi-valued fields represent the most natural tool to describe the so-called coda waves and seismic noise [18]. Indeed, due to multiple scattering on topology (or in porous systems on boundaries) plane waves are not solutions to linear field equations (for a particular topology the homogeneity of space is broken[2]). Thus if we consider any wave packet ϕ_0 it, due to multiple scattering, transforms to $\phi = \sum \phi_j$. When the topology is random, the scattering randomizes phases and such a field acquires the diffuse nature $\left\langle \phi^2 \right\rangle = \sum \left\langle \phi_j^2 \right\rangle$, i.e., each term can be

[2]The homogeneity holds only for mean statistical values.

considered as an independent field. Thus although on the micro scale the field equations remain unchanged the intensities follow a diffusion equation. The diffuse nature of seismic fields has intensively been studied (e.g., see Refs. [19] and references therein). We point out that the physical field (which is measured in experiments) represents only the sum of terms $\phi = \sum \phi_j$ and it is defined only in the physically admissible region of space. Every term however becomes an "independent field" upon a continuation to the whole coordinate space. In quantum theory particles which are described by the diffuse fields obey the generalized statistics (in particular, the violation of the Pauli principle in such fields has a rather clear physical sense; the violation occurs due to the existence of "mirror" particles in non-physical regions of space, while upon restriction to the fundamental domain the statistics restores) [16].

For the sake of simplicity and to make the basic ideas clear (and to avoid usual technical problems in quantum gravity) we, in the present paper, consider the most simple example of a set of scalar fields in R^4. The metric is supposed to be everywhere flat, while the topology is described by some gluing procedure along some multi-connected hypersurfaces. We point out that in general when considering the universal covering such gluing leads to δ- like singularities in the scalar curvature which rigorously speaking require to account for the gravitational action. To avoid such problem we shall suppose that every hypersurface is approximated by piecewise flat surfaces. Then the δ- like terms in the curvature are concentrated on vertexes and ribs which have zero measure and do not contribute to the geodesic flow. Moreover such terms possess both signs (depending on the induced curvature on the hypersurfaces) and for sufficiently complex topologies the vanishing of the mean curvature is actually not restrictive. In considering the standard continuation (by the image method) the metric remains everywhere flat, while the scattering on the topology is completely described by the bias of sources and we need not to add the gravitational action.

2 The universal covering and the topological bias

The universal covering for an arbitrary non-trivial topology of space can be constructed as follows. We take a point O in our space $\mathcal{M}$ and issue geodesics (straight lines) from O in every direction. Then points in $\mathcal{M}$ can be labeled by the distance from O and by the direction of the corresponding geodesic. In other words, for an observer at O the space $\mathcal{M}$ will always look as R^4. However if we take a point $P \in \mathcal{M}$, there may exist many homotopically non-equivalent geodesics connecting O and P. Thus, any source at the point P will have many images in R^4. The topology of $\mathcal{M}$ can be determined by noticing that in the observed space R^4 there is a fundamental domain $\mathcal{D}$ such that every point in $\mathcal{D}$ has a number of copies outside $\mathcal{D}$. The actual manifold $\mathcal{M}$ is then obtained by identifying the copies. In this way, we may describe the topology of space $\mathcal{M}$ by indicating for each point $r \in R^4$ the set of its copies $E(r)$, i.e. the set of points that are images of the same point in $\mathcal{M}$.

Consider now the actual Green function for a scalar wave equation in the physically admissible region $\mathcal{D}$

$$\left(-\Box_x + m^2\right) G\left(x, y\right) = 4\pi\delta\left(x - y\right),$$

where $x,\ y \in \mathcal{D}$. Upon continuing to the universal covering R^4 this equation transforms as follows

$$\left(-\Box_x + m^2\right) G\left(x, y\right) = 4\pi N\left(x, y\right), \tag{3}$$

where coordinates x, y are extended to the whole space R^4 and

$$N(x, y) = \delta\left(x - y\right) + \sum \delta\left(x - f_i(y)\right) \tag{4}$$

(the sum is here taken over all images of the point y, i.e., over all $f_i(y) \in E(y)$). The two point function $N\left(x, y\right)$ was called the topological bias in Refs. [11, 20] which describes the discrepancy between the actual physical space (the fundamental domain $\mathcal{D}$) and the universal covering (the simple topology space) R^4. We point out that the topology is completely (one-to-one) defined by the specifying the bias $N\left(x, y\right)$ (4). In quantum gravity (when topology may fluctuate) the bias becomes an operator valued function which has the meaning of the density of extra images for the actual source $\delta\left(x - y\right)$.

The structure of the bias (4) on the universal covering has one important feature which allows it to mimic dark matter phenomena (which are discussed in detail in Refs. [12, 11]), i.e.,

$$\int_V N(x, y) d^4x = N\left(V\right) \geq 1$$

where V is some volume around the point y. The number $N\left(V\right) - 1 = 0, 1, 2, ...$ gives the number of points $f_i(y)$ which get into the coordinate volume V. Roughly, this number characterizes how many times the volume V covers the fundamental domain (or the physically admissible region) $\mathcal{D}$.

Let us return to the path integral (2). Consider a particular virtual topology of space. It is clear that the action in (1)-(2) has the same value for all physical spaces which can be obtained by rotations and transitions of the coordinate system in R^4. Thus, upon averaging out over possible orientations and transitions the bias acquires always the structure $N\left(x, y\right) = N\left(|x - y|\right)$ and for the Green function we find

$$G\left(x - y\right) = \int \frac{d^4k}{\left(2\pi\right)^4} \frac{N\left(k\right)}{k^2 + m^2} \exp\{ik\left(x - y\right)\}, \tag{5}$$

where $N\left(k\right)$ is the Fourier transform for the bias. The above Green function plays the most important role in particle theory and its UV (ultra-violet) behavior (actually that of the bias $N\left(k\right)$) defines whether the resulting quantum theory is finite or not. What we expect that the proper definition of the path integral over virtual topologies should fix the specific form of the bias $N\left(k\right)$.

We also point out that the universal covering is what we actually use in astrophysics when extrapolating our laboratory coordinate system to extremely large distances. Therefore, in expressions (4) (5) the coordinates x, y have the direct physical (observational) status in applying to cosmological problems (DM and dark energy phenomena, origin of density perturbations etc.). In particular, we can never say (without additional subtle effects) if two points x_1 and x_2 are close or not (at least there are no external safe rulers to measure the distances). On the contrary, in high energy physics we use an extrapolation to very small scales (by means of our "safe" laboratory rulers). Again we cannot say if two points x_1 and x_2 are close or not. However we still can assign specific distances extrapolated from the laboratory coordinate system and this is exactly the coordinate system we use in particle physics. As we shall see the extrapolation in particle physics leads to the same expressions (3), (5) however the bias (4) acquires somewhat different features[3]. By other words the Universe looks somewhat different when we look at small or large distances.

3 Topological bias in particle physics

In the present section we consider the bias which originates from a single wormhole. Such a bias was constructed first in Ref. [10] for the massless field in 3-dimensions, while the generalization to the euclidean 4-space is straightforward. We point out that a wormhole describes a virtual baby universe which may branch off and joint onto our mother Universe [3]-[7].

A single wormhole can be viewed as a couple of conjugated spheres $S_\pm$ of the radius a and with a distance $d = \left|\vec{R}_+ - \vec{R}_-\right|$ between centers of spheres. The interior of the spheres is removed and surfaces are glued together. For the sake of simplicity we consider the massless case i.e., the Green function $\triangle G(x,y) = 4\pi\delta(x-y)$ for such a topology. In Ref. [10] we have shown that the proper boundary conditions (the actual topology) can be accounted for by adding the bias of the source

$$\delta(x-y) \to \delta(x-y) \ + b\left(x,y\right)$$

where in the approximation $a/d \ll 1$ the bias in R^3 takes the form

$$b\left(x\right) \approx a\left(\frac{1}{R_-} - \frac{1}{R_+}\right)\left[\delta(\vec{x}-\vec{R}_+) - \delta(\vec{x}-\vec{R}_-)\right] \tag{6}$$

where we set $y=0$ and neglect the throat size, i.e., all additional sources (ghost images) are placed in the centers of spheres. The generalization to the space R^4 is trivial and gives

$$b\left(x\right) = a^2\left(\frac{1}{R_-^2} - \frac{1}{R_+^2}\right)\left[\delta(\vec{x}-\vec{R}_+) - \delta(\vec{x}-\vec{R}_-)\right]. \tag{7}$$

[3] As it was shown in Ref. [20] in this case the bias $N\left(x,y\right)$ represents a projection operator onto physically admissible states. This means that $(\widehat{N})^2 = \widehat{N}$ and in the basis of eigenvectors it takes the form $N\left(x,y\right) = \sum N_k f_k^*\left(x\right) f_k\left(y\right)$ with eigenvalues $N_k = 0, 1$. While on the universal covering possible eigenvalues $N_k = 0, 1, 2, ...$

We see that unlike (4) the function $b(x)$ has the property $\int b(x)d^4x = 0$ which gives $\int N(x)d^4x \equiv 1$ and for any volume V we get $N(V) \leq 1$.

Let us introduce the probability distribution for parameters of the wormhole $P(R_\pm, a)$ which is defined by the action in (1). It is clear that due to homogeneity an isotropy of R^4 this function may depend only on $d = \left|\vec{R}_+ - \vec{R}_-\right|$ and we find for the mean bias

$$\overline{b}(r) = 2\int \left(\frac{1}{R^2} - \frac{1}{r^2}\right) f\left(\left|\vec{R} - \vec{r}\right|\right) d^4\vec{R}, \tag{8}$$

where $f(d) = \int a^2 P(d,a)\, da$. For the Fourier transforms $b(k) = (2\pi)^{-2} \int b(r)$ $e^{-ikr} d^4r$ this expression takes the simplest form

$$\overline{b}(k) = \frac{8\pi \left(f(k) - f(0)\right)}{k^2}. \tag{9}$$

In the so-called long-wave approximation (the low energy physics) we can completely neglect the throat size $a \to 0$. In this limit the action for the wormhole does not depend on the separation distance $d = \left|\vec{R}_+ - \vec{R}_-\right|$ at all, i.e., $P(d,a) = P(a)$, and the mean bias reduces merely to $\overline{b}(x) = b\delta(x)$ (e.g., see Ref. [10]). Therefore, the effect of wormholes reduces merely to a renormalization of physical constants (e.g., of charge values) which is in the complete agreement with the previous results of Refs. [4, 5, 7]. Moreover, the value $b < 0$ [10] which means that virtual wormholes always diminish charge values as it was first pointed out in Ref. [7].

In conclusion of this section we point out that the multiplier $4\pi/k^2$ in (9) and $1/R_\pm^2$ in (8) is the standard Green function for R^4. In the case of massive particles it should be replaced with $4\pi/\left(k^2 + m^2\right)$ and $-\frac{m^2}{8\pi z} H_1^{(2)}(z)$ (where $H_1^{(2)}(z)$ is the second order Hunkel function and $z^2 = -m^2 R_\pm^2$) respectively. Thus, we see that in particle physics the structure of the Green functions (3), (5) remains the same, while the property of the bias for the universal covering $N(V) \geq 1$ changes drastically to $N(V) \leq 1$. This feature reflects the two possible different ways of the continuation of the physical space $\mathcal{M}$ to the coordinate space R^4 (e.g., see discussions in Ref. [10]). We recall that in particle physics the bias $N(x,y)$ can be considered as a projection operator onto physically admissible states (e.g., see Sec.2 in Ref. [20]), which means that it always has eigenvalues $N_i = 0, 1$.

4 Multi-valued fields and generalized statistics

The structure of the bias (4) suggests the analogous decomposition of the true Green functions

$$G(x,y) = G_0(x-y) + \sum G_0(x - f_i(y)) \tag{10}$$

where $G_0(x-y)$ is the standard Green function for the euclidean space R^4. If we present it in the form of the path integral for a scalar particle in $\mathcal{D}$, i.e.,

$G(x,y) = \sum_{x(s)\in\mathcal{D}} \exp\left(-\int_y^x ds\right)$ then every term in (10) corresponds to the restriction of trajectories $x(s)$ to a particular homotopic class [11]. When we continue such terms to the whole space R^4 they acquire the character of independent fields that is to say that such particles has to be described by a scalar field ϕ which acquires the multi-valued (diffused) nature. An equivalen representation for such a field can be achieved in terms of the generalized statistics (e.g., see for details Ref. [16]). For the sake of convenience we present in the present section basic elements of the generalized statistics and generalized second quantization suggested first by H.S. Green. It is remarkable that the basic motivation for the generalized quantization method used by H.S. Green was the presence of mathematical inconsistencies (renormalization and regularization procedures) in quantum field theories. In the present paper we demonstrate that the goal (the removal of the inconsistencies) is actually reached.

Consider a system of identical particles with an undefined a priori symmetry of wave functions. We shall use the Bogoliubov's method [21], in which the second quantization is applied to the density matrix (to the case of para-statistics this approach was extended in Ref. [22]). Let us define operators M_{ij} of transitions for particles from a quantum state j into a state i. These operators must obey the Hermitian conditions, i.e.

$$M_{ik}^{+} = M_{ki} \tag{11}$$

and the algebra $SU(N\to\infty)$ (N is the number of different one–particle quantum states)

$$[M_{ij}, M_{km}] = \delta_{jk}M_{im} - \delta_{im}M_{kj}. \tag{12}$$

These conditions give the algebraic expression of the indistinguishability principle for identical particles. For Bose and Fermi statistics they first were established by N.N. Bogoliubov [21] and generalized to the case of arbitrary statistics by A.B. Govorkov [23].

Consider now systems with a variable number of particles. To this end we need to consider a set of creation and annihilation operators for particles (a_i^{+} and a_k) and somehow express via them the transition operators N_{ij}. The simplest generalization of Bose and Fermi statistics was first suggested by H.S. Green [17] and latter by D.V. Volkov [24] and are called the parastatistics or the Green-Volkov statistics.

Consider a set of creation and annihilation operators of particles a_i^{+} and a_k, while the transition operators we present in the form

$$M_{ik} = \frac{1}{2}\left(a_i^{+}a_k \pm a_k a_i^{+} \mp N_{ik}\right), \tag{13}$$

where N_{ik} is, in general, an arbitrary Hermitian matrix. The upper sign stands for the generalized Bose statistics, while the lower sign stands for the Fermi statistics. The operator $M_i = M_{ii}$ has sense of the particle number operator in the quantum state i. Then the creation and annihilation operators should obey the requirements

$$[M_i, a_k] = -\delta_{ik}a_k, \qquad \left[M_i, a_k^{+}\right] = \delta_{ik}a_k^{+}. \tag{14}$$

Consider now a unitary transformation

$$a'_i = \sum_k u_{ik} a_k, \qquad a'^{+}_i = \sum_k u^*_{ik} a^+_k, \tag{15}$$

where $\sum_m u_{im} u^*_{km} = \delta_{ik}$, under which the relations (12) and (14) remain invariant. Then applying to (14) an infinitesimal transformation $u_{ik} = \delta_{ik} + \varepsilon_{ik}$, $\varepsilon^*_{ik} = -\varepsilon_{ki}$ and retaining the first order terms in ε we get the basic commutation relations for the creation and annihilation operators

$$[M_{kl}, a^+_m] = \delta_{lm} a^+_k, \qquad [M_{lk}, a_m] = -\delta_{lm} a_k, \tag{16}$$

which were first suggested by Green [17].

Consider now the vacuum state $|0\rangle$ that is

$$a_k |0\rangle = 0 \tag{17}$$

for all k. Then the requirement that for all i and k the transition operators annihilate the vacuum state

$$M_{ik} |0\rangle = 0 \tag{18}$$

leads to the condition on one-particle quantum states in the form

$$a_k a^+_i |0\rangle = N_{ik} |0\rangle , \tag{19}$$

which means that the basis of one-particle states is, in general, not orthonormal but has norms $\langle 0| a_k a^+_i |0\rangle = N_{ik}$. From the physical standpoint this signals up the presence of some degeneracy of quantum states (e.g., the presence of an extra coordinate etc., see discussions in Refs. [16]).

The fact that N_{ik} is a Hermitian matrix denotes that there exists a basis of one-particle wave functions in terms of which this matrix has the diagonal form, i.e., $N_{ik} = \delta_{ik} N_k$. H.S. Green (and after him all other invesigators) imposed the Lorentz invariance on the form of Eq. (19) which imediately gives the simplest case $N_k = N$ (where N is a constant), i.e., the form $N_{ik} = \delta_{ik} N$ remains invariant in an arbitrary basis[4].

The condition that norms of vectors in the Fock space are positively defined leads to the requirement that N is an integer number which characterizes the rank of the statistics or the rate of the degeneracy of quantum states [17, 22]. In the simplest case the number N corresponds to the maximal number of particles which admit an antisymmetric (for parabosons and symmetric for parafermions) state. The case $N = 0$ corresponds to the absence of fields. The case $N = 1$ corresponds to the standard Bose and Fermi statistics.

[4]It is easy to see the analogy of the above matrix N_{ik} with the bias $N(k, k')$ introduced previously. This analogy indicates the existence of a very deep relation between these two operators. We point out that the bias reflects the discrepancy between the topology of the actual physical space and that of R^4. Therefore the Lorentz invariance imposed on Eq. (19) immediately kills the baby. Since any particular topology breaks the Lorentz invariance, it may hold only for mean values.

For the case of a constant rank Green also presented an ansatz which resolves the relations (16) and (19) in terms of the standard Bose and Fermi creation and annihilation operators

$$a_p^+ = \sum_{\alpha=1}^{N} b_p^{(\alpha)+}, \quad a_k = \sum_{\alpha=1}^{N} b_k^{(\alpha)}, \tag{20}$$

where $b_p^{(\alpha)}$ and $b_p^{(\beta)+}$ are the standard Bose (Fermi) operators as $\alpha = \beta$ ($\left[b_p^{(\alpha)} b_k^{(\alpha)+}\right]_\pm = \delta_{pk}$) but anti-commutate (commutate) as $\alpha \neq \beta$ ($\left[b_p^{(\alpha)} b_k^{(\beta)+}\right]_\mp = 0$) for the case of parabose (parafermi) statistics. The presence of an additional index α in the creation and annihilation operators removes the degeneration of one-particle quantum states pointed out.

The Green representation (20) can be easily generalized to the more general case of an arbitrary Hermitian matrix N_{ik}. In the basis in which this matrix takes the diagonal form $N_{ik} = N_k \delta_{ik}$ the Green representation is given by the same expression (20) in which, however, the rank of statistics N_k depends on the quantum state (the index k). Thus, in the general case the rank of statistics represents an additional quantum variable. We also note that in an arbitrary basis the Green representation does not work and, therefore, we can say that the matrix N_{ik} distinguishes a preferred basis of quantum states, see also discussions in Refs. [16].

5 Multi-valued fields and the action

In the case of a homogeneous and isotropic topological structure the multi-valued character of the scalar field is more convenient to describe in the Fourier representation ($\phi = \frac{1}{(2\pi)^2} \int d^4k \phi_k e^{ikx}$) that is to replace the single-valued field ϕ_k with a set of fields ϕ_k^j where $j = 1, 2, ...N(k)$, while the bias $N(k)$ has the meaning of the number of such fields (or the rank of statistics). We recall that from the phenomenological standpoint such fields were introduced first in Ref. [14] and for the relation to the generalized statistics see Refs. [16].

Consider now the euclidean action for the scalar field (we use the Planckian units in which $M_{pl} = 1$)

$$S = \frac{1}{2} \int \left[(\partial_\mu \phi)^2 + m^2 \phi^2 + V(\phi) \right] d^4x. \tag{21}$$

Rigorously speaking the integral here should run only over the fundamental domain $\mathcal{D}$. However to describe different possible topologies on an equal footing we should continue this expression on the whole space R^4. In what follows we shall use the Fourier transform for the field, while the actual topology will be encoded by specifying $N(k)$ (we assume that the integration over transitions and orientations in (2) is already carried out and therefore $N(k)$ defines a whole

class of topologies, while S is the modified action). Then the linear part of the action takes the structure[5]

$$S_0 = \frac{L^4}{2} \int \sum_{j=1}^{N(k)} \left(k^2 + m^2\right) \left|\phi_k^j\right|^2 \frac{d^4k}{(2\pi)^4}, \tag{22}$$

while the non-linear term $S_{int}(\phi)$ should be accounted for by perturbations. We recall that in this expression the values of the number of fields $N(k)$ depend on scales under consideration and, therefore, the result for the cutoff function depends on the choice of the continuation used. As it was explained previously in astrophysical problems we use the universal covering and the number of fields takes values $N(k) = 0, 1, 2, ...$, while in particle physics the number of fields can take only two possible values $N(k) = 0, 1$.

The physical sense has the sum of fields, and therefore the generating functional should be taken as

$$\begin{aligned} \widetilde{Z}[J] &= \exp\left\{-S_{int}\left(\frac{\delta}{\delta J}\right)\right\} \int D[\phi] \exp\left\{-S_0(\phi) + L^4 \int J(-k)\,\widetilde{\phi}_k d^4k\right\} \\ &= \widetilde{Z}[0] \exp\left\{-S_{int}\left(\frac{\delta}{\delta J}\right)\right\} \exp\left\{\frac{L^4}{2} \int \frac{|J(k)|^2}{k^2+m^2} N(k) \frac{d^4k}{(2\pi)^4}\right\} \end{aligned} \tag{23}$$

where $\widetilde{\phi}_k = \sum_{j=1}^{N(k)} \phi_k^j$, while for $\widetilde{Z}[0]$ we find

$$\widetilde{Z}[0] = \exp\left\{-\frac{L^4}{2} \int N(k) \frac{d^4k}{(2\pi)^4} \ln \frac{k^2+m^2}{\pi}\right\}. \tag{24}$$

In particular, we can write $\widetilde{Z}[0] = \exp\left(-L^4 < \rho >_{eff}\right)$, where $< \rho >_{eff}$ is the zero-point vacuum energy density which for a particular topology $N(k)$ is

$$< \rho >_{eff} = \frac{1}{2} \int N(k) \frac{d^4k}{(2\pi)^4} \ln \frac{k^2+m^2}{\pi}. \tag{25}$$

Thus we see whether the cosmological constant is finite or not depends on the topological structure of the actual space. Now to account for all possible virtual topologies (spacetime foam) and get the final expression for the generating function $Z[J]$ we have to sum over topologies, i.e., possible values of $N(k)$ in accordance to (2). For sure we may expect that all topologies which give infinite values of $< \rho >_{eff}$ should be suppressed.

[5] We point out that such a simple form for the linear part of the action is reached only for isotropic and homogeneous class of topologies, while for a particular topology the bias has the structure $N = N(k, k')$ and the action diagonalizes in a specific (for given topology) basis e.g., see discussions in Refs.[16, 20].

6 Cutoff function in particle physics

While the topology is fixed, $N(k)$ is an ordinary fixed function[6]. Now we are ready to evaluate the cutoff for the particle physics in the case when topology may fluctuate. In this case possible values of $N(k)$ are 0 and 1. The partition function (24) has the structure

$$\widetilde{Z}[0] = \prod_k Z_k^{N(k)}$$

where Z_k is given by the standard single-field expression $Z_k = \sqrt{\pi / (k^2 + m^2)}$ and the sum over possible values $N(k)$ gives

$$Z = \sum_{topologies} \widetilde{Z}[0] = \prod_k \left(\sum_{N=0,1} Z_k^{N(k)} \right) = \prod_k (1 + Z_k), \tag{26}$$

while for the mean cutoff we find from (1)

$$\overline{N}(k) = \frac{Z_k}{(1 + Z_k)}. \tag{27}$$

This expression straightforwardly generalizes on a multiplet of scalar fields or a set of bosonic fields of an arbitrary spin which gives

$$\ln Z_k = \frac{1}{2} \sum_\alpha \ln \frac{\pi}{(k^2 + m_\alpha^2)}, \tag{28}$$

where the sum is taken over all fields and helicity states. The bias and the cutoff for Fermi fields require a separate consideration which we consider elsewhere[7].

The remarkable property of the cutoff function is the explicit Lorentz invariance (i.e., the function $\overline{N}(k)$ depends on the momenta via the Lorentz invariant expression k^2). On the mas-shell $Z_k \to \infty$ and it reduces to $N(k) \to 1$ which reflects the fact that on the mas shell the space looks as R^4, while at very small (planckian) scales $Z_k \ll 1$ it has the behavior $N(k) \sim 1/k^g \to 0$ as $k \to \infty$, (where g is the total number of degrees of freedom). Thus, as it was expected [1] for sufficiently big number of fields g, $N(k)$ provides indeed a Lorentz invariant cutoff which we discuss in the next section.

[6] Actually $N(k)$ defines the whole topological class, while a specific topology is fixed by a function $N(k, k')$.

[7] Standard fermionic fields have the negative energy in the ground state which leads to the instability of fermionic fields with respect to topology fluctuations. However, we point out that the action for fermionic fields has no the classical limit and therefore it is defined up to a constant shift (the cosmological constant term). Moreover, if we consider some coarse graining in the phase space, then the difference between fermions and bosons should disappear (upon the coarse graining, more than one fermion can occupy the same quantum state). By other words the instability pointed out should lead to a phase transition upon which fermionic excitations acquire positive energy density in the ground state and become stable with respect to topology fluctuations.

7 Finiteness of Feynman diagrams

The generating functional $\widetilde{Z}[J]$ leads to the standard perturbation scheme (e.g., see the standard textbooks [25]). New features however appear. As we can see from (5) and (23) the integration measure for every closed loop takes the form $N(k)\, d^4k/(2\pi)^4$ and, therefore, every diagram will include the factor $\langle N(k_1)\, N(k_2) \ldots N(k_n)\rangle$ which in the first only approximation by topology fluctuations can be replaced with the product $\overline{N}(k_1)\,\overline{N}(k_2)\ldots\overline{N}(k_n)$ where $\overline{N}(k)$ gives the cutoff which is defined by (27). Thus every Feynman diagram acquires an additional decomposition onto a series by topology fluctuations of the cutoff function.

The contribution in the cutoff function $\overline{N}(k)$ comes from all physical fundamental fields (28) and it is clear that all UV divergencies are automatically regularized (e.g., if we account only for gravitational $h_{\mu\nu}$, electromagnetic A_ν, and weak Z_ν, $W_\nu^{\pm}$ interactions, the number of degrees of freedom is 10 and it defines the UV behavior $N(k) \sim 1/k^{10}$ as $k \to \infty$ which is already sufficient to regularize all divergent diagrams[8]. In (27) the characteristic UV scale of the cutoff has the planckian order $Z_k \sim 1$, which means that Z_k includes contribution of all fields with mas less than planckian mas m_{pl}. This is not convenient for practical computations; for the actual cutoff occurs for much lower energies. To see this let us introduce the characteristic scale $k \sim \mu$ which has the sense of the laboratory scale from which we extrapolate our laboratory coordinate system to very small distances (i.e. the actual scale of the cutoff). From the analogy with the statistical physics such a scale can be viewed as a specific chemical potential which corresponds to the additional cosmological constant term to the action[9], i.e., the redefinition of (25) as

$$< \rho >_{eff} = \frac{1}{2}\int N(k)\,\frac{d^4k}{(2\pi)^4}\ln\frac{k^2+m^2}{\mu^2}. \tag{29}$$

Then Z_k modifies as $Z_k \to Z_k/Z_\mu$ and the cutoff function (27) modifies as

$$\overline{N}(k) = \frac{Z_k}{(Z_\mu + Z_k)}. \tag{30}$$

In such a form we may retain in Z_k only the necessary (smallest) number of fields with masses $m_\alpha < \mu$, while all more massive particles give only a constant contribution to $Z_k \sim \mu/m$ and lead merely to a renormalization of the scale μ itself. By other words we may suppose that the contribution of the most heavy particles is already encoded in μ (at least this allows also to account phenomenologically for all possible new particles and fields which may be found in the future at extremely high energies).

[8] We point out that gauge fields have more components whose contribution to Z_k depends on the choice of the gauge fixing. Therefore the exponent in $N(k) \sim 1/k^g$ may be even more than ten.

[9] We recall that when we consider interactions all constants acqire a dependence on scales [25].

Thus the cutoff function acquires the structure

$$\overline{N}\left(k\right)=\frac{\mu^{g}}{\left(\mu^{g}+k^{2\alpha_{0}}\left(k^{2}+m_{1}^{2}\right)^{\alpha_{1}}\cdots\left(k^{2}+m_{n}^{2}\right)^{\alpha_{n}}\right)} \tag{31}$$

where $m_\alpha < \mu$ and $g = \sum 2\alpha_n$ is the total number of fields we have to retain.

The most divergent expressions in quantum field theory come from terms of the type $\left\langle (\partial\phi)^2 \right\rangle$, which in the momentum space have UV behavior[10] $\sim p^4$. We point out that p^4 gives also the highest rate of divergency in quantum gravity as well, e.g. see Ref. [26]. As an example of such a term we consider the cosmological constant (29). Since all fields which we retain in (31) give some contribution to the cosmological constant $<\rho>_{eff}$ we sum (29) over all fields which gives (upon simple transformations)

$$<\rho>_{eff}=\frac{\mu^{4}}{\left(16\pi^{2}\right)}F\left(\alpha,\widetilde{m}\right), \tag{32}$$

where

$$F\left(\alpha,\widetilde{m}\right)=\int_{0}^{\infty}\frac{\ln\left(x^{\alpha_{0}}\left(x+\widetilde{m}_{1}\right)^{\alpha_{1}}\cdots\left(x+\widetilde{m}_{n}\right)^{\alpha_{n}}\right)}{\left(1+x^{\alpha_{0}}\left(x+\widetilde{m}_{1}\right)^{\alpha_{1}}\cdots\left(x+\widetilde{m}_{n}\right)^{\alpha_{n}}\right)}xdx$$

and $\widetilde{m}_i = m_i^2/\mu^2$. This expression is finite for $\sum 2\alpha_n > 4$ (i.e., we have to retain at least five field degrees of freedom). In the case when $\widetilde{m}_i = 0$ it gives

$$F\left(\alpha,0\right)=-\frac{\pi^{2}}{\alpha_{0}}\frac{\cos\left(2\pi/\alpha_{0}\right)}{\sin^{2}\left(2\pi/\alpha_{0}\right)}.$$

Next "dangerous" terms are given by $\left\langle \phi^2 \right\rangle$ which define the renormalization of the mas. We evaluate it for $\lambda\phi^4$ [25] which in the first order by λ gives the correction to the mas (the so-called "tadpole" diagram)

$$\delta m^{2}=\Sigma\left(p\right)=\frac{\lambda}{2}\int N\left(k\right)\frac{d^{4}k}{\left(2\pi\right)^{4}}\frac{1}{k^{2}+m^{2}} \tag{33}$$

which gives

$$\Sigma\left(p\right)=\Sigma\left(0\right)=\frac{\lambda}{32\pi^{2}}\mu^{2}G\left(\alpha,\widetilde{m}\right),$$

where

$$G\left(\alpha,\widetilde{m}\right)=\int_{0}^{\infty}\frac{xdx}{\left(x+\widetilde{m}\right)\left(1+x^{\alpha_{0}}\left(x+\widetilde{m}_{1}\right)^{\alpha_{1}}\cdots\left(x+\widetilde{m}_{n}\right)^{\alpha_{n}}\right)}$$

which is already finite for $\sum 2\alpha_n > 2$. In the massless case it gives

$$G\left(\alpha,0\right)=\frac{1}{\alpha_{0}}\Gamma\left(1/\alpha_{0}\right)\Gamma\left(1-1/\alpha_{0}\right).$$

[10] Actually the most divergent behavior will be given by $\sim p^8$, when fluctuations in the cutoff function itself are taken into account, since the Gaussian character of the distribution over $N(k)$ gives $\overline{\Delta N^2} \sim \overline{N}$. However such terms should be treated in the complete analogy with the subsequent analysis.

In this manner we see that all divergencies in Feynman diagrams disappear when the contribution of a proper number of fields in the cutoff function is taken into account. It is quite clear that this result is valid almost in all theories (whose dynamical equations do not include too high derivatives of fields which in general lead to p^n divergencies) and it seems to remain true in general relativity (GR) as well [26]. However unlike gauge fields (which are proved to be renormalizable) GR represents formally non-renormalizable theory[11] and therefore it requires the more complete and rigorous proof which we leave for the future research.

8 Cutoff function on the universal covering

In observational cosmology when we look at the sky we always use the coordinate system which corresponds to the universal covering. Therefore, in solving astrophysical problems (quantum origin of density perturbations, quantum cosmology, etc.) we have to use the representation in which the actual space is described by the universal covering. We recall that in general the universal covering requires the introducing of a curved background and, therefore, the results of the present section have only a preliminary character.

In the present section we evaluate the astrophysical cutoff function as well. In this case the number of fields takes the values $N(k) = 0, 1, 2, ...$ and (26) becomes

$$Z = \sum_{topologies} \widetilde{Z}[0] = \prod_k \left(\sum_{N=0}^{\infty} \frac{Z_k^{N(k)}}{N(k)!} \right) = \exp \left(L^4 \int Z_k \frac{d^4k}{(2\pi)^4} \right), \qquad (34)$$

where we have accounted for the fact that permutations of fields at the same k gives the same quantum state (i.e., the identity of fields which gives the factor $1/N!$). Then for the mean cutoff we find from (1)

$$\overline{N}(k) = Z_k \ . \qquad (35)$$

Thus (27) and (35) define the relation between the bias (cutoffs) in the two different representations for the same physical space.

The analogy with the statistical physics shows that (34) (35) correspond to the classical (or the Boltzmann) statistics. As it was discussed in the introduction such statistics corresponds to the so-called diffused fields [19]. However quantum topology should introduce some additional statistics between fields [14] which corresponds to third quantization and which we consider in what follows[12].

Consider first the density of fields in the configuration space (i.e., the space of fields)

$$N[k, \phi] = \sum_j \delta \left(\phi - \phi_k^j \right)$$

[11] There is only a small chance that due to the entanglement in complex diagrams divergencies may remain.

[12] Such correlations may be important in investigating corrections to the mean values of the type $\langle N(k_1) N(k_2) ... N(k_n) \rangle$ which appear in Feynman diagrams.

so that the number of fields is merely

$$N(k) = \int N[k,\phi]\, d\phi.$$

Then the action (22) can be rewritten as

$$S = \frac{L^4}{2} \int N[k,\phi] \left(k^2 + m^2\right) |\phi|^2 \, D\phi \frac{d^4k}{(2\pi)^4}$$

which represents the functional of $N[k,\phi]$. Thus, the partition function can be presented as

$$Z = \sum_{N[k,\phi]} \exp\left\{-S\left(N[k,\phi]\right)\right\}.$$

Here the sum over $N[k,\phi]$ includes, in fact, both the sum over topologies and configuration variables. The further depends on the statistics of fields assumed (which is not the same as the statistics of particles, e.g., see Refs. [14, 16]). If we accept the Fermi statistics (i.e., numbers $N[k,\phi] = 0, 1$) then such scalar particles will obey the so-called para-Bose statistics [16]. The choice should be made from experiment (though there may be some theoretical reasoning for a particular choice). In both cases we find for the mean density

$$\langle N[k,\phi]\rangle = \left[\exp\left(\frac{1}{2}\left(k^2 + m^2\right)|\phi|^2\right) \pm 1\right]^{-1}$$

and for the cutoff function we find the same expression (35) with an additional multiplier

$$\overline{N}(k) = C_{\pm} Z_k^g$$

where the multiplier is given by (g is the number of components of the scalar field ϕ)

$$C_{\pm} = \frac{1}{\pi^{g/2}} \int \frac{d^g\phi}{\exp\left(\frac{1}{2}|\phi|^2\right) \pm 1}.$$

9 Conclusions

In conclusion we briefly repeat basic results. First of all we have explicitly demonstrated that spacetime foam provides quantum fields with a cutoff. The form of the cutoff is fixed by the field theory itself and it does not introduce additional parameters. It depends only on the standard set of naked parameters related to fields. It does also depend on the representation of the physical space used. We have to used the two types of different representations depending on the problem under consideration. In particle physics we extrapolate the laboratory coordinate system to extremely small scales and, therefore, we should use the so-called standard representation (the image method) which gives (31) for the cutoff. In astrophysics however we always have deal with the universal

covering and the cutoff becomes (35). Since we considered quantum topology fluctuations around the flat space, the cutoff has the Lorentz invariant form. This is always justified for particle physics, while in the astrophysical picture our results carry rather a preliminary character; for rigorous consideration requires a curved background. In the present Letter our consideration has a simplified character, i.e., a set of scalar fields. However it is clear that all the results can be straightforwardly extended to any non-linear field theory. In particular, the cutoff suggested automatically regularizes divergencies in quantum fields and, therefore, we can expect that general relativity represents in fact a renormalizable theory.

We also demonstrated that every Feynman diagram acquires an additional decomposition onto a series by topology fluctuations in the cutoff function which may lead to some new phenomena.

The cutoff function has the meaning of the topological bias of point sources which displays the discrepancy between the visual and the actual spaces. In astrophysics such a discrepancy is observed as the Dark Matter phenomenon [11, 20]. Analogous phenomena are widely known in particle physics which represent "Dark Charges" of all sorts. Those are not more than the standard (phenomenological) Higgs fields [27]. Therefore, we expect that quantum gravity provides the unique tool to fix all constants of nature (the lambda term, mas spectrum, charge values, etc.). However the self-consistent evaluation of such parameters requires considering the complete theory which is to be developed.

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